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IFToMM Benchmark Problem
Linearized Bicycle

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Chapter 1

Introduction

This report provides the results of the analysis of the IFToMM linear bicycle benchmark problem using the EoM software produced by the University of Windsor Vehicle Dynamics and Control research group. The problem consists of four rigid bodies connected by three revolute joints, and two rolling contacts. The properties are summarized below. The problem is modified slightly by defining a steering torque between the fork and frame as the system input and the steer and lean angles as the system output. Note that there may be some round-off in the data describing the system properties in this document, but that the full precision values were used to define the system. Please see the problem definition document for the more precise values.

1.1 System Description

The properties of the bodies are given in Tables 1.1 and 1.2. The properties of the connecctions are given in Table 1.3.

Table 1.1: Body CG Locations and Mass

No.	Body Name	Location [m]	Mass [kg]
1	frame	0.300, 0.000, −0.900	85.000
2	fork	0.900, 0.000, −0.700	4.000
3	front-wheel	1.020, 0.000, −0.350	3.000
4	rear-wheel	0.000, 0.000, −0.300	2.000

Table 1.2: Body Inertia Properties

No.	Body Name	Inertia [kg·m^2] (I_{xx}, I_{yy}, I_{zz}; I_{xy}, I_{yz}, I_{zx})
1	frame	9.200, 11.000, 2.800; 0.000, 0.000, −2.400
2	fork	0.059, 0.060, 0.007; 0.000, 0.000, 0.008
3	front-wheel	0.141, 0.280, 0.141; 0.000, 0.000, 0.000
4	rear-wheel	0.060, 0.120, 0.060; 0.000, 0.000, 0.000

Note: inertias are defined as the positive integral over the body, e.g., $I_{xy} = +\int r_x r_y \, dm$.

Table 1.3: Connection Location and Direction

No.	Connection Name	Location [m]	Unit Axis
1	head	0.853, 0.000, −0.761	0.309, 0.000, 0.951
2	rear axle	0.000, 0.000, −0.300	0.000, 1.000, 0.000
3	front axle	1.020, 0.000, −0.350	0.000, 1.000, 0.000
4	rear road	0.000, 0.000, 0.000	0.000, 1.000, 0.000
5	front road	1.020, 0.000, 0.000	0.000, 1.000, 0.000
6	speed	0.300, 0.000, −0.900	1.000, 0.000, 0.000
7	front tire	1.020, 0.000, 0.000	0.000, 1.000, 0.000
8	rear tire	0.000, 0.000, 0.000	0.000, 1.000, 0.000

Chapter 2

Analysis

The EoM software automatically conducts a linear analysis after producing the linearized equations of motion. The results are shown in Figure 2.1, and show strong agreement with the IFToMM benchmark results.

One area of discrepancy is the presence of additional zero eigenvalues. In the benchmark definition, the longitudinal, lateral, and yaw motions are treated as ignorable coordinates, and do not appear in the equations of motion. In the formulation used by the EoM software, neutrally stable motions are included in the result, and so a number of additional zero eigenvalues appear. In a comparison of the EoM results to the benchmark, the zero eigenvalues show by far the largest error, on the order of 1×10^{-6}, while the the non-zero eigenvalues have error on the order of 1×10^{-13}. If desired, with relatively little additional computational effort, the state space formulation can be reduced to a minimal realization that eliminates the ignorable modes from the equations of motion, leaving only the non-zero eigenvalues.

The numerical values of the eigenvalues generated by the EoM software are included in a separate text file.

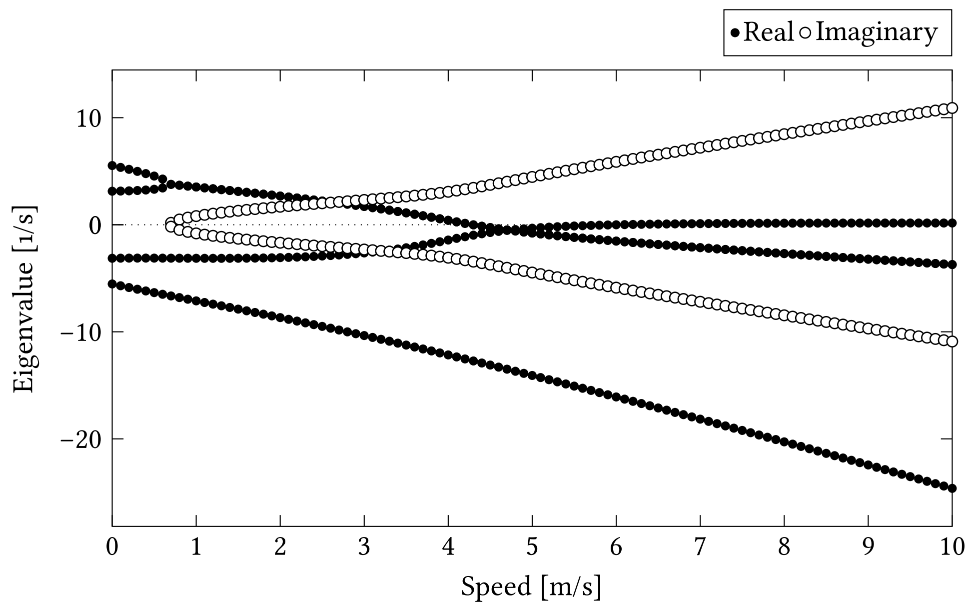

Figure 2.1: Eigenvalues vs. Speed

Appendix A

Equations of Motion

The equations of motion are prepared in first order form.

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & -\mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{w}} \\ \dot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \mathbf{V} & -\mathbf{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{L} & -\mathbf{F} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{w} \\ \boldsymbol{u} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u} \end{Bmatrix}$$

The state vector consists of six global position coordinates for each body $\boldsymbol{p}$, and six body fixed velocity coordinates $\boldsymbol{w}$ for each body. The input vector $\boldsymbol{u}$ is appended to the state. The first row of the system are the linearized kinematic differential equations. The matrix $\mathbf{V}$ is determined only by the velocity around which the linearization occurs. The second row of the system are the linearized Newton Euler equations. The matrix $\mathbf{M}$ represents the mass and inertia terms, the matrix $\mathbf{G}$ allows the inclusion of systems that have dependency on the rate of input. The $\mathbf{K}$ is the stiffness matrix, and includes terms that depend on the physical stiffnes, the geometry, and the static preload carried in the connections. The matrix $\mathbf{L}$ is the damping matrix, and depends on the physical damping coefficients, the geometry, and the velocity of linearization, in order to include the centripetal and gyroscopic terms. The $\mathbf{F}$ matrix represents the senitvity to the various inputs. The bottom row appends the inputs to the state vector.

The system is subject to constraints.

$$\begin{bmatrix} \mathbf{J}_{\mathrm{h}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{J}_{\mathrm{h}}\mathbf{V} & \mathbf{J}_{\mathrm{h}} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\mathrm{nh}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} & \boldsymbol{p} \\ \dot{\boldsymbol{w}} & \boldsymbol{w} \\ \dot{\boldsymbol{u}} & \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The first row of the system are the holonomic contraints, applied to position and rate of change of position. The $\mathbf{J}_{\mathrm{h}}$ matrix is the constraint Jacobian. The second row of the system are also the holonomic contraints, applied to velocity and rate of change of velocity. There is redundancy between the first entry in the first row and the second entry in the second row. The third row of the system are the nonholonomic contraints, applied to velocity and rate of change of velocity.

A null space of the constraint matrix is found, and used to reduce the system to a smaller set of coordinates, in the vector $\boldsymbol{x}$, and giving the matrices $\mathbf{E}$, $\mathbf{A}$, and $\mathbf{B}$. The result is combined with a set of output equations, also transformed to minimal coordinates, to give **C** and **D**. Note that the resulting system may still define a set of DAEs, depending on the condition of the resulting **E** matrix. If **E** is singular, then the system of equations can be further reduced to a minimal realization.

The full state space equations:

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{Bmatrix}$$